Wavelets and Multifractal Formalism for Singular Signals: Application to Turbulence Data

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The wavelet decomposition is used to generalize the multifractal formalism to singular signals. The singularity spectrum is directly determined from the scaling behavior of partition functions that are defined from the wavelet transform modulus maxima. Illustrations on fractal signals with a recursive structure, e.g., devil's staircases, are shown. Applications to fully developed turbulence data and Brownian signals are reported.

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The recently developed multifractal formalism [1] has proven particularly fruitful in the characterization of singular measures arising in a variety of physical situations [2]. This formalism accounts for the statistical scaling properties of these measures through the determination of their singularity spectrum [1] $f(\alpha)$, which is intimately related to the generalized fractal dimensions [3] D_q. The $f(\alpha)$ singularity spectrum provides a rather intuitive description of a multifractal measure in terms of interwoven sets of Hausdorff dimension $f(\alpha)$ corresponding to singularity strength α. Actually, the concept of multifractality originated from a general class of multiplicative cascade models introduced by Mandelbrot [4(a)] in the context of fully developed turbulence. Measurements of the $f(\alpha)$ spectrum based on the local dissipation have recently brought conspicuous experimental evidence for the multifractal nature of the dissipation field in turbulent flows [5]. An alternative multifractal description [6] of the intermittency of the fine structures consists of extracting the spectrum $D(h)$ of Hölder exponents h of the velocity field from the inertial scaling properties of structure functions: $S_p(l) = \langle(\delta v_l)^p\rangle \sim l^{\zeta_p}$ (p integer > 0), where δv_l is a longitudinal velocity increment over a distance l. $D(h)$ is essentially the Legendre transform of the scaling exponents ζ_p. There are, however, some fundamental drawbacks to this method. Indeed, it generally fails to fully characterize the singularity spectrum $D(h)$, since only the strongest singularities are amenable to this analysis. Even though one can extend this study from integer to real positive p values by considering an absolute value on velocity increments, the structure functions generally do not exist for $p < 0$. Moreover, singularities of Hölder exponents $h > 1$ and regular behavior introduce drastic bias in the estimate of the ζ_p's. The purpose of this Letter is to elaborate on a novel approach which will allow us to determine the whole singularity spectrum $D(h)$ directly from any experimental signal. This strategy is based on the use of a new tool introduced in signal analysis, *the wavelet transform* [7] (WT), which has proven very powerful in characterizing the scaling properties of multifractal measures [8]. Beyond the reported applications, our ambition is actually to establish the foundations for a thermodynamical formalism for singular signals [9].

The WT of a signal permits an analysis both in physical space and in scale space [7]. It consists in expanding functions in terms of *wavelets* which are constructed from one single function, the analyzing wavelet g, by means of dilations and translations. The WT of a signal $s(x)$ is defined as

$$T_g(a,x_0) = \frac{1}{a}\int_{-\infty}^{+\infty} s(x)\bar{g}\left(\frac{x-x_0}{a}\right)dx, \quad a>0. \tag{1}$$

Provided g is well localized around $x=0$ and has a vanishing integral, this transformation is invertible for a large class of signals s. The WT can be used as a mathematical microscope [8] to analyze the local regularity of functions [10]. In fact, if the signal $s(x)$ has, at the point x_0, a local scaling (Hölder) exponent $h(x_0)$ in the sense that, in the limit $l \to 0$,

$$s(x_0+l) = s(x_0) + ls'(x_0) + \cdots + (l^n/n!)s^{(n)}(x_0) + C|l|^{h(x_0)} + o(|l|^{h(x_0)})$$

$[n < h(x_0) < n+1]$, then $T_g(a,x_0) \sim a^{h(x_0)}$ for $a \to 0$, provided the first $n+1$ moments of g are zero. Thus one can extract the exponent $h(x_0)$, for fixed position x_0, from a log-log plot of the WT amplitude versus the scale a. The situation is somewhat more intricate when investigating fractal signals due to the existence of a hierarchical distribution of singularities [2]. Locally the Hölder exponent $h(x_0)$ is then governed by the singularities which accumulate at x_0. This results in unavoidable oscillations around the expected power-law behavior of the WT amplitude. The exact determination of h from log-log plots on a finite range of scales is therefore somewhat uncertain. In particular the local scaling behavior is not well defined when these oscillations are nonperiodic or periodic with period larger than the investigated range of scales [8,11]. Of course there have been many attempts to circumvent these difficulties, e.g., to follow in the (a,x) half plane a curve of WT modulus maxima [12,13] emanating from x_0 or to adapt locally the shape of the analyzing wavelet [14], in order to reduce the oscillation amplitude. In some circumstances, ergodic formulas have been established by which the Hölder exponents can be obtained as zoom averages over logarithmically varying scales [11]. Nevertheless, there exist fundamental limitations (which are not intrinsic to the WT technique) to the measure of Hölder exponents from local scaling behavior in a finite range of scales. Therefore the determination of statistical quantities like the singularity spectrum $D(h)$

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requires a method which is more feasible and more appropriate than a systematic investigation of WT local scaling behavior as previously experienced in Ref. [13(b)].

The classical multifractal description [1] in terms of thermodynamic quantities supposes, more or less explicitly, the existence of an underlying multiplicative process [4,15]. Previous applications [8] of the wavelet microscope to multifractal measures lying on Cantor sets have demonstrated its fascinating ability to reveal the construction process (renormalization operation [8(b)]) of recursive singular measures. Since the analyzing wavelet g can be chosen orthogonal to polynomial behavior of arbitrarily high order, the WT capability to capture the intricate singularity arrangement is likely to extend to fractal signals. As proven by Mallat and Hwang [12], the WT modulus maxima [i.e., the local maxima of $|T_g(a,x)|$ at a given scale a] detect all the singularities of a large class of signals. Thus they are likely to contain all the information on the hierarchical distribution of singularities in the signal. This is clearly illustrated in Fig. 1(a) where the positions of the WT modulus maxima of the devil's staircase shown in Fig. 2(a) reveal the construction rule of the uniform triadic Cantor set upon which are located the singularities of this continuous and almost everywhere differentiable signal. At the scale $a = a_0 3^{-n}$, each one of the $k_0 2^n$ modulus maxima simultaneously bifurcates into two new maxima giving rise to a cascade of symmetric pitchfork branchings in the limit $a \rightarrow 0$. Our method of computing the singularity spectrum of a fractal signal will consist in taking advantage of the space-scale partitioning given by the maxima representation to define a partition function which scales, in the limit $a \rightarrow 0$, in the following way:

$$Z(a,q) = \sum_{\{x_i(a)\}_i} |T_g(a,x_i(a))|^q \sim a^{\tau(q)} . \qquad (2)$$

At a given scale a, by not summing over the whole set of wavelet coefficients but only over the WT modulus maxima $\{x_i(a)\}_i$, we directly incorporate the multiplicative structure of the singularity distribution into the calculation of the partition function (for a generalization to an adaptive scale partitioning, see Ref. [9(b)]). Moreover, in doing so, we get rid of divergences for negative q values. Let us notice that $\tau(0) = -D_F$; thus, the fractal dimension D_F can be seen as the ratio of the logarithms of the average maxima multiplication rate and the average scale factor, respectively. More generally, if one identifies in Eq. (2), $|T_g(a,x_i)|$ with its asymptotic power-law behavior $a^{h(x_i)}$, in the limit $a \rightarrow 0$, using the method of steepest descent, one can show that the singularity spectrum $D(h)$ is the Legendre transform of $\tau(q)$. But computing the Legendre transform has several disadvantages that may lead to various errors [5]. An alternative method is to define $D(h)$ in the spirit of the so-called canonical method defined in Ref. [5(b)]:

$$h(q) = \lim_{a \rightarrow 0} \frac{1}{\ln a} \sum_{\{x_i(a)\}_i} \hat{T}_g(q;a,x_i(a)) \ln |T_g(a,x_i(a))| , \qquad (3a)$$

$$D(h(q)) = \lim_{a \rightarrow 0} \frac{1}{\ln a} \sum_{\{x_i(a)\}_i} \hat{T}_g(q;a,x_i(a)) \ln \hat{T}_g(q;a,x_i(a)) , \qquad (3b)$$

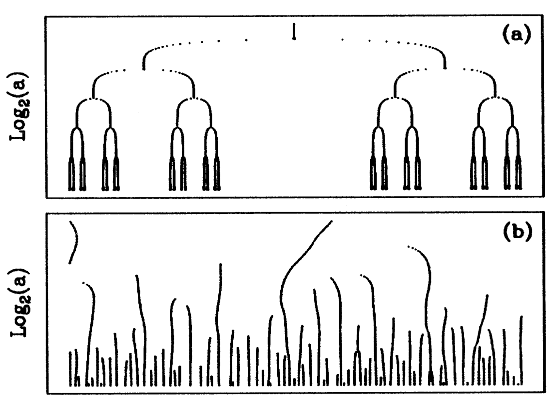

FIG. 1. WT skeleton showing the positions of the modulus maxima of T_g for (a) the devil's staircase and (b) the turbulent velocity signal. In (a) and (b) g is the first and the second derivative of the Gaussian function, respectively.

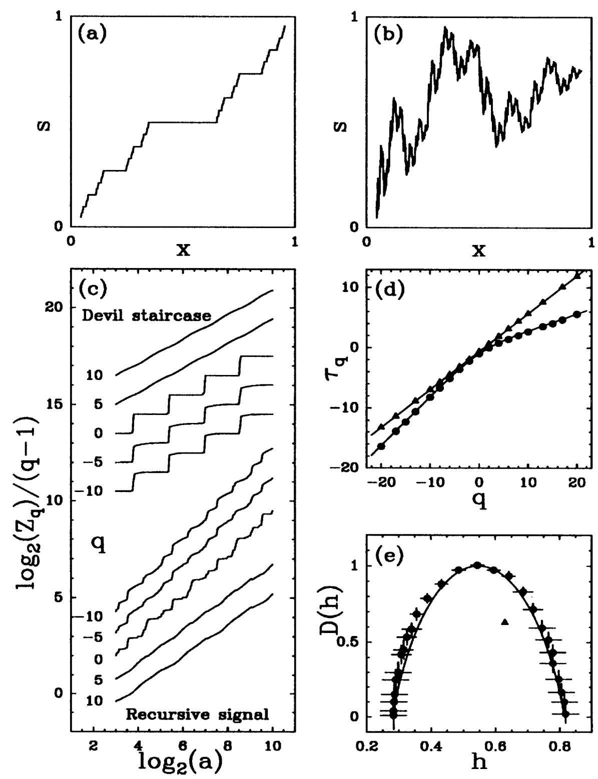

FIG. 2. WT measurement of the singularity spectrum of (a) the devil's staircase and (b) of a multifractal signal with a deterministic recursive structure. g is the second derivative of the Gaussian function. (c) $\log_2 Z(a,q)/(q-1)$ vs $\log_2 a$. (d) $\tau(q)$ vs q. (e) $D(h)$ vs h. In (d) and (e), ▲, devil's staircase; ●, multifractal recursive signal; and ——, theoretical predictions.

where $\hat{T}_g(q;a,x_i) = |T_g(a,x_i)|^q / \sum_{x_i} |T_g(a,x_i)|^q$. One can thus directly extract the set of Hölder exponents h and the corresponding $D(h)$ spectrum from log-log plots of the quantities in Eqs. (3). At this point, let us mention the main pitfall to our method which is essentially based on the tracking of WT modulus maxima. There may exist extra maxima in our wavelet representation that do not correspond to any singularity in the signal. A way to remove these spurious maxima would consist of working with an analyzing wavelet which has all its moments equal to zero. Unfortunately these wavelets are not of practical use since they oscillate too much and thus lead to a proliferation of maxima which make our partition function calculation not workable numerically. For each application reported below, we have used real analyzing wavelets $g^{(n)}$ among the class of derivatives of the Gaussian function [7]. The robustness of our method has been tested when increasing the order n of derivation.

As illustrated in Figs. 2(c) and 2(d), the scaling exponents $\tau(q)$ of $Z(a,q)$ display a characteristic linear behavior as a function of q when considering monofractal signals like the devil's staircase in Fig. 2(a). The slope of the graph $\tau(q)$ provides an accurate estimate of the unique Hölder exponent $h = \ln 2/\ln 3$. By either Legendre transforming $\tau(q)$ or using Eqs. (3), one gets $D(h = \ln 2/\ln 3) = \ln 2/\ln 3$, i.e., the fractal dimension of the uniform triadic Cantor set. On more general grounds, one can prove rigorously [9(b)] that for distribution functions $s(x) = \int_{-\infty}^{x} d\mu(x)$ of some invariant measures of expanding Markov maps [1(b)] ("cookie cutter" Cantor sets), $\tau(q) = (q-1)D_q$, where D_q is the q-order generalized fractal dimensions of μ and $D(h)$ is the Hausdorff dimension of the set of singularities of Hölder exponent h of s. For this class of signals, the cascade of WT modulus maxima is conservative in the sense that $\tau(1) = 0$ as a consequence of the normalization of μ at each scale. In Fig. 2 are also shown the results of a similar analysis for a multifractal signal [Fig. 2(b)] constructed according to a more general deterministic recursive process. As found in Fig. 2(d), $\tau(q)$ is an increasing convex nonlinear function of q. Its Legendre transform $D(h)$ in Fig. 2(e) is a well-defined unimodal curve, the support of which extends over a finite interval $h_{\min} \le h \le h_{\max}$. The maximum of this curve $D(h(q=0)) = -\tau(0) = 1$ gives the fractal dimension of the support of the set of singularities of s; s is almost everywhere singular. Generally $\tau(1) \ne 0$, which indicates that the cascade of WT modulus maxima is not conservative. In Figs. 2(d) and 2(e) the numerical data are compared with the theoretical predictions; the agreement for $-20 \le q \le 20$ is quite remarkable. As seen in Fig. 2(c), there exist oscillations superimposed on the power-law behavior (2) of the partition function. The main difference from the oscillations observed in the WT local scaling behavior is the fact that these oscillations are periodic [discrete scaling invariance of $Z(a,q)$], with a period which corresponds to one step in the multiplicative recursive construction process [9]. Provided the accessible range of scales contain a few oscillation periods, one can thus extract the scaling exponents $\tau(q)$ with a good accuracy. Similar results have been obtained for stochastic signals generated with a random multiplicative process [9].

In recent years there have been several attempts to apply the wavelet analysis to fully developed turbulence data [13,16]. A preliminary investigation of the velocity field at inertial range scales has revealed a multifractal branching process in the wavelet representation that is clearly different from the fractal branching observed with Gaussian processes [13(a)]. Dynamically significant events having strong localized gradients have been identified in the turbulent signal [13(b)]. However, because of the operational limitations of the local scaling analysis, there has been thus far no reliable determination of the singularity spectrum $D(h)$ directly from the velocity signal without recourse to dissipation-type quantities. In Figs. 1(b) and 3, we present the preliminary results of our analysis of a turbulent velocity signal recorded in the wind tunnel S1 of ONERA at Modane [17]. The Taylor-scale-based Reynolds number is $R_\lambda = 2720$ and the extent of the inertial range is almost three decades. The results reported here concern the analysis in the inertial range of about 100 integral length scales of the recorded turbulent signal. The analysis of a fractional Brownian signal [4(b)] having a $k^{-5/3}$ spectrum is shown for comparison. When plotted versus q, the scaling exponents $\tau(q)$ obtained for the fractional Brownian motion remarkably fall on a line of slope $h = 0.33 \pm 0.01$ [Fig. 3(b)]. The monofractality of the Brownian signal is

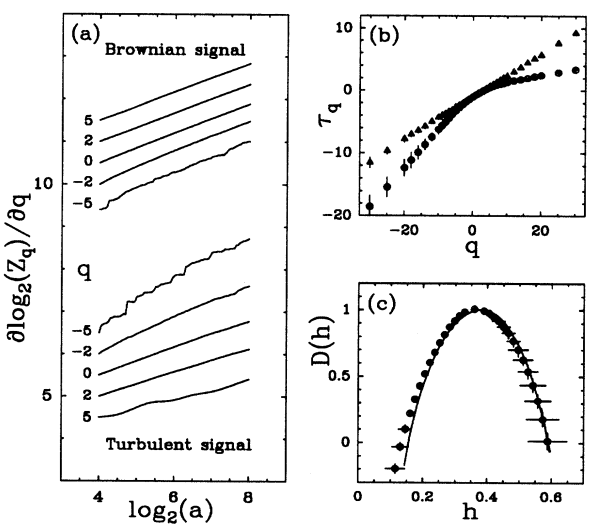

FIG. 3. WT measurement of the singularity spectrum of fractional Brownian motion and turbulent velocity signal [17]. g is as in Fig. 2. (a) Log-log measurements of $h(q)$ from Eq. (3a). (b) $\tau(q)$ vs q. (c) $D(h)$ vs h. In (b) and (c), ▲, Brownian signal; ●, turbulent signal; and ——, average singularity spectrum obtained from dissipation field data [5] via the Kolmogorov scaling relation.

confirmed in Fig. 3(a) where the direct estimate of $h(q)$ from Eq. (3a) does not reveal any significant q dependence of this exponent. Similarly, from Eq. (3b) one gets $D(h) = 1.00 \pm 0.01$. Thus, as expected theoretically [18], we find that the Brownian signal is almost everywhere singular with a unique Hölder exponent $h \simeq \frac{1}{3}$. In Fig. 3(b), the $\tau(q)$ curve extracted from the WT modulus maxima [Fig. 1(b)] of the turbulent signal unambiguously deviates from a straight line. The values of h obtained when varying q from -30 to 30 range in the interval [0.11,0.60] [Fig. 3(a)]. The corresponding singularity spectrum $D(h)$ [Fig. 3(c)] displays the characteristic single-humped shape of multifractal signals. Its maximum value $D(h(q=0)) = -\tau(0) = 1.00 \pm 0.01$ strongly suggests that the turbulent signal is almost everywhere singular [9]. The manifest part [4(c)] of $D(h)$ (>0) is compared to a solid curve which actually corresponds to a common fit of short-term and long-term statistics data of dissipation fields at lower Reynolds numbers [5]. This curve has been deduced from the experimental average $f(\alpha)$ spectrum of the energy dissipation ϵ by using the Kolmogorov scaling relation [19] $\epsilon_l \sim (\delta v_l)^3/l$. The fact that one cannot discriminate between these two singularity spectra within the experimental uncertainty can be interpreted *a posteriori* as an experimental verification of the Kolmogorov hypothesis. This observation can also be understood as an experimental confirmation of the universality of the multifractal singularity spectrum of fully developed turbulence [5]. However, it is clear that considerable further work is needed for reliable quantitative conclusions. In particular long-term statistical analysis must be carried out in order to capture more accurately the latent part [4(c)] [$D(h) < 0$] of the singularity spectrum including possible violent rare events corresponding to negative Hölder exponents [13(b)]. This analysis is currently in progress.

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